

## ON SCHUR-SZEGÖ COMPOSITION OF POLYNOMIALS

VLADIMIR KOSTOV AND BORIS SHAPIRO

**ABSTRACT.** Schur-Szegő composition of two polynomials of degree less or equal than a given positive integer  $n$  introduces an interesting semigroup structure on polynomial spaces and is one of the basic tools in the analytic theory of polynomials, see [4]. In the present paper we add several (apparently) new aspects to the previously known properties of this operation. Namely, we show how it interacts with the stratification of polynomials according to the multiplicities of their zeros and present the induced semigroup structure on the set of all ordered partitions of  $n$ .

The Schur-Szegő composition of two polynomials  $P(x) = \sum_{i=0}^n C_n^i a_i x^i$  and  $Q(x) = \sum_{i=0}^n C_n^i b_i x^i$  is given by  $P * Q(x) = \sum_{i=0}^n C_n^i a_i b_i x^i$ , see e.g. [5]. Let  $Pol_n$  denote the linear space of all polynomials in  $x$  of degree at most  $n$ . In what follows we always use its standard monomial basis  $\mathcal{B} := (x^n, x^{n-1}, \dots, 1)$ . To any polynomial  $P \in Pol_n$  one can associate the operator  $T_P$  which acts diagonally in  $\mathcal{B}$  and is uniquely determined by the condition:  $T_P(1+x)^n = P(x)$ . Obviously, for  $P(x) = C_n^0 a_0 + C_n^1 a_1 x + \dots + C_n^n a_n x^n$  one has  $T_P(x^i) = a_i$ ,  $i = 0, 1, \dots, n$ . Given  $P$  as above we refer to the sequence  $\{a_i\}$  as to the *diagonal sequence* of  $P$ . Any two such operators  $T_P$  and  $T_Q$  commute and their product  $T_P T_Q$  corresponds in the above sense exactly to the Schur-Szegő composition  $P * Q$ . The famous composition theorem of Schur and Szegő (see original [5] and e.g. §3.4 of [4] or §2 of [1]) reads:

**Theorem 1.** *Given any linear-fractional image  $K$  of the unit disk containing all the roots of  $P$  one has that any root of  $P * Q$  is the product of some root of  $Q$  by  $-\gamma$  where  $\gamma \in K$ .*

Geometric consequences of Theorem 1, in particular, Proposition 2, can be found in § 5.5 of [4]. A polynomial  $P \in Pol_n$  is called *hyperbolic* if all its roots are real. Denote by  $Hyp_n \subset Pol_n$  the set of all hyperbolic polynomials and by  $Hyp_n^+ \subset Hyp_n$  (resp.  $Hyp_n^- \subset Hyp_n$ ) the set of all hyperbolic polynomials with all positive (resp. all negative) roots. Denote by  $H_{u,v,w} \subset Hyp_n$  (where  $u, v, w \in \mathbf{N} \cup 0$ ,  $u+v+w = n$ ) the set of all hyperbolic polynomials with  $u$  negative and  $w$  positive roots and a  $v$ -fold zero root.

*Proposition 2* (Theorem 5.5.5 and Corollary 5.5.10 of [4]). *If  $P, Q \in Hyp_n$  and if  $Q \in Hyp_n^+$  or  $Q \in Hyp_n^-$ , then  $P * Q \in Hyp_n$ . Moreover, all roots of  $P * Q$  lie in  $[-M, -m]$  where  $M$  is the maximal and  $m$  is the minimal pairwise product of roots of  $P$  and  $Q$ .*

A diagonal sequence, (or an operator  $T : Pol_n \rightarrow Pol_n$  acting diagonally in  $\mathcal{B}$ ) is called a *finite multiplier sequence (FMS)*, see [3], if it sends  $Hyp_n$  into  $Hyp_n$ . The set  $\mathcal{M}_n$  of all FMS is a semigroup. For the following characterization of FMS see [2], Theorem 3.7 or [1], Theorem 3.1.

**Theorem 3.** *For  $T \in \text{End}(Pol_k^{\mathbb{R}})$  the following two conditions are equivalent:*

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$(-1)^i C_n^i b_i$ , one has  $K_j = C_j^i$  for  $j \geq i$ ,  $K_j = 0$  for  $j < i$ , the left side equals  $\sum_{j=i}^n (-1)^j C_n^j C_j^i \sum_{\nu=0}^{n-j} C_{n-j}^\nu b_{\nu+j}$  (\*\*) and one checks directly that the coefficient before  $b_l$  in (\*\*) equals  $(-1)^i C_n^i$  if  $l = i$  and 0 if  $l \neq i$ .) Hence, if  $\mu^* > 0$ , then  $-1$  is a root of  $P * Q$  – each product in the left side of (\*) contains a zero factor. When  $\mu^* = 0$ , then all but one products contain such a factor, so  $(P * Q)(-1) \neq 0$ . To prove that  $-1$  is a root of  $P * Q$  of multiplicity  $\mu^*$  one has to show for  $\lambda < \mu^*$  that  $\sum_{j=0}^{n-\lambda} (-1)^j C_{n-\lambda}^j K_j^\lambda L_{n-\lambda-j} = \sum_{j=0}^{n-\lambda} (-1)^j C_{n-\lambda}^j a_{j+\lambda} b_{j+\lambda} = \frac{(n-\lambda)!}{n!} (P * Q)^{(\nu)}(-1)$  where  $K_j^\lambda = \sum_{q=0}^j C_j^q a_{q+\lambda} = \frac{j!}{n!} (x^{n-\lambda} P^{(\lambda)}(1/x))^{(n-\lambda-j)}|_{x=1}$ .  $\square$

*Proof of Proposition 5:* We prove it in the case  $v = 0$ , i.e for any  $P \in \text{Hyp}_n$ ,  $P(0) \neq 0$  and any  $Q \in \text{Hyp}_n^-$ . The general case follows by continuity. The statement is trivially true for any hyperbolic  $P(x)$  of degree  $n$  and  $Q(x) = (1+x)^n$  since  $P * Q = P$ . Let  $Q \in \text{Hyp}_n^-$ . Connect  $Q(x)$  to  $(1+x)^n$  by some path  $Q^t(x)$  within  $\text{Hyp}_n^-$ . (This is possible since  $\text{Hyp}_n^-$  is contractible.) Notice that if  $P(0) \neq 0$ , then  $P * Q^t(0) \neq 0$  for the whole family since the constant term of  $P * Q$  is the product of the ones of  $P$  and  $Q$ . Therefore the number of positive and negative roots of  $P * Q$  is the same as for  $P * (1+x)^n$ .  $\square$

*Proof of Theorem 6(i):* Instead of  $(P * Q)(x)$  we consider  $Z(x) := (P * Q)(-x)$  (to have the same ordering of the roots on the line in all three polynomials). Suppose that  $P \in H_{s,l,n-s-l}$ , i.e.  $P = (\prod_{j=1}^s (x + a_j)) x^l (\prod_{j=s+l+1}^n (x - a_j))$ ,  $a_j > 0$ . Fix  $a_j$  for  $j = 1, \dots, s$  and deform them continuously into 0 for  $j = s+l+1, \dots, n$ . The MV of the negative root sets of  $Z$  does not change. Therefore to find this MV it suffices to find it for  $P$  replaced by  $P_1 := (\prod_{j=1}^s (x + a_j)) x^{n-s}$ . In the same way, to find the MV of the positive root sets of  $Z$  it suffices to find it for  $P$  replaced by  $P_2 := x^{s+l} (\prod_{j=s+l+1}^n (x - a_j))$ . When  $P(x)$  is changed to  $P(-x)$ , then  $Z(x)$  changes to  $Z(-x)$ , (this explains the presence of  $\text{sign}(\alpha)$  in (i)) and the description of the MV of the root sets of  $Z$  can be done by considering only polynomials of the form  $P_2$ .

Consider the case when  $P, Q \in \text{Hyp}_n^+$  (hence,  $P$  and  $Q$  play the same role). The other three cases  $P \in \text{Hyp}_n^\pm$ ,  $Q \in \text{Hyp}_n^\pm$  can be treated by analogy using  $P(-x) * Q(x) = P(x) * Q(-x) = (P * Q)(-x)$ . If  $P = (x - a)^n$ , then  $Z(x) = Q(ax)$ , so assume that each polynomial  $P, Q$  has two distinct roots,  $0 < a_1 < a_2$  and  $0 < b_1 < b_2$ , of multiplicities  $m_1, m_2$  and  $n_1, n_2$ . If  $n$  is even and  $m_1 = n_1 = n/2$ , then  $Z$  has no A-roots. Recall that by Proposition 4 if  $a_i b_j$  is an A-root, then its multiplicity is  $m_i + n_j - n$ .

Assume that (one of) the biggest of the four multiplicities  $m_1, m_2, n_1, n_2$  is among the last two. Suppose first that this is  $n_1$ . If  $n_1 + m_1 > n$ ,  $n_1 + m_2 > n$ , then the root set  $R(Z)$  of  $Z$  looks like this:  $(a_1 b_1, V, a_2 b_1, Y)$ , see Propositions 4 and 2. Set  $\sharp(V) = v$ ,  $\sharp(Y) = y$ . When writing  $b_1 \rightarrow 0$  or  $b_1 \rightarrow b_2$  we mean that the roots  $a_1, a_2, b_2$  are fixed. When  $b_1 \rightarrow 0$ , then in the limit  $Z$  has  $n_2$  non-zero roots which are all from  $Y$ , hence,  $y \geq n_2$ . When  $b_2 \rightarrow b_1$ , then in the limit  $Z(x) = P(b_1 x)$  has two roots, of multiplicities  $m_1$  and  $m_2$ . Hence,  $(n_1 + m_1 - n) + v \geq m_1$ , i.e.  $v \geq n_2$ . But  $v + y = 2n_2$ , hence,  $v = y = n_2$ .

If  $n_1 + m_1 > n \geq n_1 + m_2$ , then  $R(Z) = (a_1 b_1, V)$ ,  $v = n_2 + m_2$ . If  $n_1 + m_1 \leq n < n_1 + m_2$ , then  $R(Z) = (U, a_2 b_1, V)$ ,  $u + v = n_2 + m_1$ . When  $b_1 \rightarrow 0$ , then  $v \geq n_2$  because all  $n_2$  non-zero (in the limit) roots are in  $V$ . When  $b_2 \rightarrow b_1$ , then  $(n_1 + m_2 - n) + v \leq m_2$ , i.e.  $v \leq n_2$ . Hence,  $v = n_2$ ,  $u = m_1$ .

Let  $n_2 = \max(m_1, m_2, n_1, n_2)$ . If  $n_2 + m_1 > n$ ,  $n_2 + m_2 > n$ , then  $R(Z) = (U, a_1 b_2, V, a_2 b_2)$ . When  $b_1 \rightarrow 0$ , this yields  $u \geq n_1$ , and  $b_1 \rightarrow b_2$  yields  $v \geq n_1$ . As  $u + v = 2n_1$ , one has  $u = v = n_1$ . If  $n_2 + m_1 > n \geq n_2 + m_2$ , then  $R(Z) = (U, a_1 b_2, V)$ . When  $b_1 \rightarrow 0$ , this yields  $u \geq n_1$ , and  $a_1 \rightarrow 0$  implies  $v \geq m_2$ . As

$u + v = m_2 + n_1$ , one has  $u = n_1$ ,  $v = m_2$ . If  $n_2 + m_1 \leq n < n_2 + m_2$ , then  $R(Z) = (U, a_2 b_2)$ ,  $u = m_1 + n_1$ . This proves (i) of Theorem 6 for  $P, Q \in \text{Hyp}_n^+$  having each  $\leq 2$  distinct roots.

Further we assume that  $P$  has a single A-root  $a$ , of multiplicity  $m$ . To prove the theorem by induction on the number of distinct positive roots in  $P$  and  $Q$  it suffices to consider the result on the MV of the root sets of  $Z$  when a multiple root of  $P$  or  $Q$  splits into two. If this is a B-root, such a splitting deforms continuously the B-roots in  $Z$ , its A-roots and their multiplicities don't change, and the theorem holds.

If an A-root splits into two B-roots, then it is a root of  $Q$ . Suppose that this is  $b_{j_d}$ , and that one has  $b_{j_{d-1}} < b_{j_d} < b_{j_{d+1}}$ ,  $b_{j_d}$  being A-roots. Denote the multiplicities of these three roots by  $h_1, h_2, h_3$ , and by  $t_1, t_2$  the sums of the multiplicities of the B-roots of  $Q$  from  $(b_{j_{d-1}}, b_{j_d})$  and  $(b_{j_d}, b_{j_{d+1}})$ . Before the splitting of  $b_{j_d}$  the polynomial  $Z$  had three A-roots stemming from  $b_{j_{d-1}} < b_{j_d} < b_{j_{d+1}}$ , namely,  $ab_{j_{d-1}} < ab_{j_d} < ab_{j_{d+1}}$ , of multiplicities  $m + h_i - n$ ,  $i = 1, 2, 3$ , with sums of the multiplicities of the B-roots of  $Z$  from the two intervals between them equal to  $t_1 + n - m$ ,  $t_2 + n - m$ . After the splitting there remain only the A-roots  $ab_{j_{d-1}} < ab_{j_{d+1}}$ , the A-root  $ab_{j_d}$  splits into B-roots of total multiplicity  $m + h_2 - n$ . In  $Q$  there remain the A-roots  $b_{j_{d-1}} < b_{j_{d+1}}$  with total multiplicity of the B-roots between them equal to  $t_1 + t_2 + h_2$ . Thus the sum of the multiplicities of the B-roots of  $Z$  from  $(ab_{j_{d-1}}, ab_{j_{d+1}})$  after the splitting equals  $t_1 + t_2 - 2m + 2n + m + h_2 - n = t_1 + t_2 + h_2 + n - m$ . Hence, (i) of Theorem 6 holds after the splitting. If the A-root  $b_{j_d}$  is first or last, i.e.  $d = 1$  or  $d = r$ , then the proof is similar.

Suppose that an A-root (say,  $c$  of  $P$ , of multiplicity  $\mu$ ) is splitting into an A-root to the left and a B-root to the right, of multiplicities  $\xi$  and  $\eta$ . Then in  $Z$  there is a splitting of an A-root  $cf$  ( $f$  is an A-root of  $Q$ ) of multiplicity  $\mu + \nu - n$  into an A-root of multiplicity  $\xi + \nu - n$  and one or several B-roots of total multiplicity  $\eta$ . Suppose that at least one of these B-roots goes to the left. Shift to the left (after the splitting) all roots of  $P$  simultaneously while keeping the ones of  $Q$  fixed. When one has  $c = 0$ , then the number of positive roots (counted with the multiplicities) will be greater for  $P$  than for  $Z$  (this follows from  $[cf]_+ = [c]_+ + [f]_+$  before the splitting). This is a contradiction with Proposition 5. Hence, all new B-roots of  $Z$  go to the right after the splitting and one checks directly that (i) of Theorem 6 holds after the splitting. If the B-root of  $P$  goes to the left, or if  $c$  is a root of  $Q$ , then the reasoning is similar.

If an A-root  $c$  splits into two A-roots  $c^1$  (left) and  $c^2$  (right) (hence,  $c$  is a root of  $Q$ ), then the above reasoning shows that in  $Z$  an A-root  $cf$  splits into two A-roots  $c^1 f$  (left,  $f$  is an A-root of  $P$ ) and  $c^2 f$  (right) and one or several B-roots between them. Indeed, one shows as above that all roots different from  $c^1 f$  (resp.  $c^2 f$ ) and stemming from  $cf$  must go right (resp. left). Hence, the B-roots of  $Z$  resulting from the splitting are between  $c^1 f$  and  $c^2 f$ . Denote by  $n^0, n^1, n^2$  and  $m^0$  the multiplicities of  $c, c^1, c^2$  and  $f$  ( $n^0 = n^1 + n^2$ ). Hence, the multiplicities of  $c^1 f, c^2 f$  and the total multiplicity of the B-roots of  $Z$  between them equal  $n^1 + m^0 - n, n^2 + m^0 - n$  and  $n - m^0$ , i.e. (i) of Theorem 6 holds after the splitting.

*Proof of Theorem 6(ii):* We show that non-simplicity of a B-root contradicts  $P * Q \in \text{Hyp}_n$  for any  $P \in \text{Hyp}_n, Q \in \text{Hyp}_n^\pm$ , see Proposition 2. We first settle the basic case when either  $P$  or  $Q$  has only simple zeros and then use a procedure which either decreases the multiplicity of some root of  $P$  or leads to  $P * Q \notin \text{Hyp}_n$ . The multiplicity of 0 as a root of  $P$  must decrease up to 0, not to 1.

1) Basic case. Suppose that  $b \neq 0$  is a B-root of  $P * Q$  of multiplicity  $\mu \geq 2$ . If  $P$  has distinct real non-zero roots, then such are all polynomials from a small neighbourhood  $\Delta$  of  $P$  in  $\text{Pol}_n^\mathbb{R}$ . If  $\mu = 2$ , then adjusting the constant term which

is non-zero by assumption one can easily choose  $T \in \Delta$  such that  $T * Q$  have a complex conjugate pair of zeros close to  $b$  – a contradiction. If  $\mu > 2$ , then one can choose  $T$  such that  $(T * Q)'$  has a multiple root at  $b$  and  $(T * Q)(b) \neq 0$ . Hence,  $T * Q \notin \text{Hyp}_n$ .

2) General case. Assume that  $P = (x - c)^l P_1(x)$ ,  $l \geq 2$ ,  $P_1(c) \neq 0$ ,  $c \neq 0$ . Set  $P_c(x) := P(x)/(x - c)$ . Consider the family of hyperbolic polynomials  $P^\delta = P + \delta P_c$ ,  $\delta \in \mathbb{R}$  ( $\dagger$ ). In this family the  $l$ -tuple root  $c$  splits into the  $(l - 1)$ -tuple root  $c$  and an extra root  $c - \delta$  which is simple unless it coincides with some other root of  $P$ . Set  $U_c := P_c * Q$ . Further considerations split into 3 subcases below:

2.i) If  $U_c(b) \neq 0$ , then we find a value of  $\delta$  such that  $P^\delta * Q \notin \text{Hyp}_n$  – a contradiction. Indeed, if  $\mu$  is even, then choosing  $\text{sign}(\delta)$  one obtains that  $P^\delta * Q$  has no real roots close to  $b$ . If  $\mu$  is odd, choose its sign so that the total multiplicity of the roots of  $P * Q$  close to  $b$  is  $< \mu$  (when  $U'_c(b) \neq 0$ , then  $P * Q$  can be made monotonous close to  $b$ ; when  $U'_c(b) = 0 \neq U''_c(b)$ , then choose  $\text{sgn}(\delta)$  so that there is a local minimum (maximum) of  $P * Q$  close to  $b$  where  $P * Q$  is positive (negative); if  $U'_c(b) = U''_c(b) = 0$ , then for  $\delta \neq 0$ ,  $b$  is a degenerate critical point and a non-root of  $P * Q$ , hence,  $P * Q \notin \text{Hyp}_n$ ).

2.ii) If  $U_c(b) = U'_c(b) = 0$ , then  $P^\delta * Q$  still has a multiple B-root at  $b$  and lower multiplicity of  $c$ .

2.iii) Suppose that  $U_c(b) = 0 \neq U'_c(b)$ . Assume first that this happens for at least two distinct roots  $c, d$  of  $P$  of which  $c$  is multiple or  $c = 0$ . Set  $P_{cd} := P/((x - c)(x - d))$ . Consider the 2-parameter family of hyperbolic polynomials  $P^{\delta, \varepsilon}(x) = P(x) + \delta P_c(x) + \varepsilon P_d(x) + \delta \varepsilon P_{cd}(x)$  ( $\ddagger$ ). In this family the root  $c$  (and also  $d$  when multiple) splits as in ( $\dagger$ ). Observe that  $P_{cd} = (P_c - P_d)/(c - d)$  ( $+$ ). Hence,  $P_{cd} * Q(b) = 0$ . Set  $\delta = -\varepsilon(P_d * Q)'(b)/((P_c + \varepsilon P_{cd}) * Q)'(b)$ . For  $\varepsilon \neq 0$  small enough one has  $((P_c + \varepsilon P_{cd}) * Q)'(b) \neq 0$ . With this choice of  $\delta$  the polynomial  $P^{\delta, \varepsilon} * Q$  still has a multiple root at  $b$  and lower multiplicity of  $c$ .

2.iv) To finish the argument notice that the only case to consider when one cannot perform splittings of roots of  $P$  is when  $P$  has a single multiple or zero root  $c$  (of multiplicity  $\nu$ ) with  $U_c(b) = 0 \neq U'_c(b)$ , and the remaining non-zero roots  $d_i$  of  $P$  are all simple with  $(P_{d_i} * Q)(b) = (P_{d_i} * Q)'(b) = 0$ . The same must be true for  $Q$ ; denote by  $g$  the root of  $Q$  of multiplicity  $\lambda > 1$ . But then a suitable linear combination of  $P$  and  $P_{d_i}$  equals  $(x - c)^n$  (see ( $+$ ) etc.). Hence,  $(x - c)^n * Q = Q(cx)$  has a multiple root at  $b$ , i.e.  $b = cg$ . As  $b$  must be a B-root (and not an A-root) of  $P * Q$ , one must have  $\nu + \lambda \leq n$ . If  $\nu + \lambda = n$ , then  $b$  is a non-root of  $P * Q$  by Proposition 4 – a contradiction. If  $\nu + \lambda < n$ , then a suitable linear combination of  $P$  and  $P_{d_i}$  equals  $Y := (x - h)^\nu (x - c)^{n - \nu}$  for which one has that  $b$  is a multiple root of  $Y * Q$  – a contradiction with Proposition 4 again.  $\square$

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LABORATOIRE J.-A. DIEUDONNÉ, UMR 6621 DU CNRS, UNIVERSITÉ DE NICE, PARC VALROSE,  
F-06108 NICE, CEDEX 2

*E-mail address:* `kostov@math.unice.fr`

MATHEMATICS DEPT., STOCKHOLM UNIVERSITY, S-106 91 STOCKHOLM, SWEDEN

*E-mail address:* `shapiro@math.su.se`